Instructor:	Frank Secretain
Course:	Math 20
Assessment:	Final
Time allowed:	110 minutes
Devices allowed:	Pencil, pen, eraser, calculator
Notes from instructor:	Be neat. Show your work where needed. Box final answers.
Marks allocated:	7 questions worth 30 marks + 1 bonus question worth 2 marks
Percentage of final grade:	25% of final grade

Formula Sheet

Arithmetic Series

$$a_n = a_1 + (n-1)k$$

$$S_n = \sum_{i=1}^{n} a_1 + (i-1)k$$

$$= \frac{n}{2}(a_1 + a_n)$$

Geometric Series

$$a_n = a_1 r^{n-1}$$

$$S_n = \sum_{i=1}^{n} a_1 r^{i-1}$$

$$= a_1 \frac{1-r^n}{1-r}$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^{n-k} y^k$$

Line equation

$$y = ax + b$$

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition of the derivative

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Rules of differentiation

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$ (product rule)

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$ (quotient rule)

$$\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(f(g(x)))\frac{d}{dx}(g(x))$$ (chain rule)

Derivatives of select functions

$$\frac{d}{dx}(ax^n) = anx^{n-1}$$

$$\frac{d}{dx}(sin(x)) = cos(x)$$

$$\frac{d}{dx}(cos(x)) = -sin(x)$$

$$\frac{d}{dx}(tan(x)) = \frac{1}{(cos(x))^2}$$

$$\frac{d}{dx}(a^x) = a^x ln(a)$$

$$\frac{d}{dx}(log_a x) = \frac{1}{xln(a)}$$

Integrals of select functions

$$\int ax^n dx = \begin{cases} \frac{a}{n+1}x^{n+1} & , n \neq -1 \\ aln(|x|) & , n = -1 \end{cases}$$ (polynomials)

$$\int sin(ax)dx = -\frac{1}{a}cos(ax)$$

$$\int cos(ax)dx = \frac{1}{a}sin(ax)$$ (trigonometry)

$$\int tan(ax)dx = \frac{1}{a}ln(|sec(ax)|)$$

$$\int a^x dx = \frac{1}{ln(a)}a^x$$

$$\int ln(x)dx = xln(x) - x$$ (exponentials)

Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n$$

Integration by parts

$$\int udv = uv - \int vdu$$

Frank Secretain's math course review

name: anonymous

Overall rating of this course

1	2	3	4	5	6	7	8	9	10

(worst course ever) (best course ever)

Things you liked about the course:

Things you didn't like about the course:

Other comments:

Thank you for your comments
Have a great holiday

(2 marks) Determine the 800th number in the sequence and the sum from the first number to the 800th number for the following series.

-397, -396, -395, -394, …

(2 marks) If a material degrades to 99.8% of the previous years strength, how strong will the material be (compared to the original strength) after 100 years?

(2 marks each) Take the derivative with respect to “x” of the following functions.

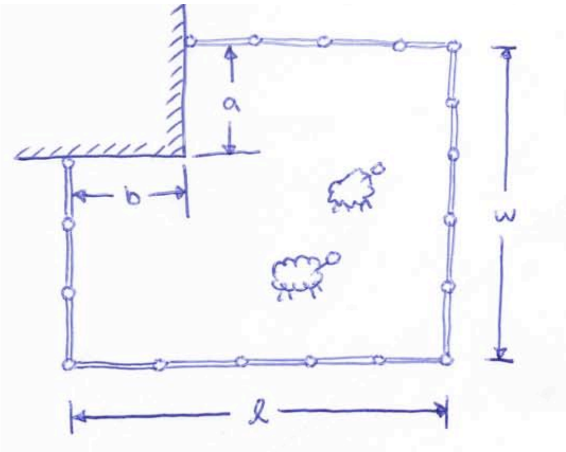
$$y(x) = 2x^3 + 5sin(2x)$$

$$y(x) = 2(x+1)^3 sin(2x)$$

$$y(x) = \lambda x^2 + \alpha \sin(2x) + \rho$$

$$V(x) = \frac{(R-r)^2\pi}{3H^2}x^3 + \frac{r\pi}{H}(R-r)x^2 + r^2\pi x$$

(5 marks) You have to build a rectangular fence keeping a corner section that will not be fenced, as shown in the diagram. If you are given “c” units of fencing what dimension would it have to maximize area?

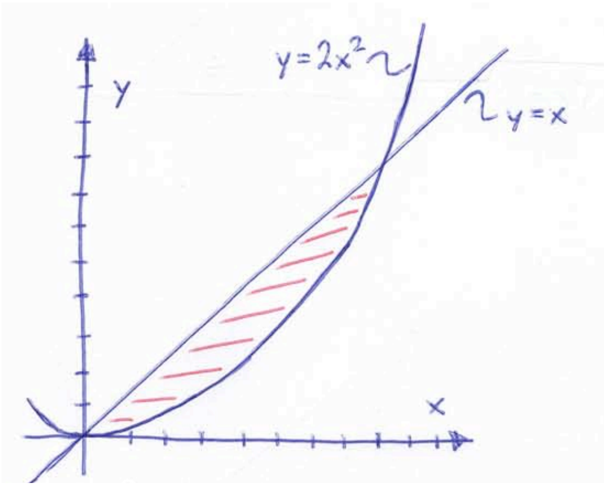


(2 marks each) Integrate with respect to “x” the following functions.

$$\int 7x^2 + 8\sin(x-1) + \alpha \quad dx$$

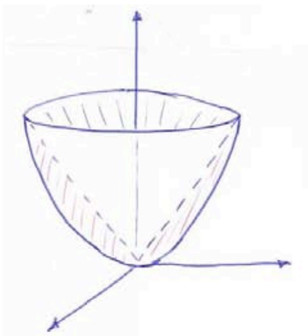
$$\int 2x\sqrt{x^2+1} \quad dx$$

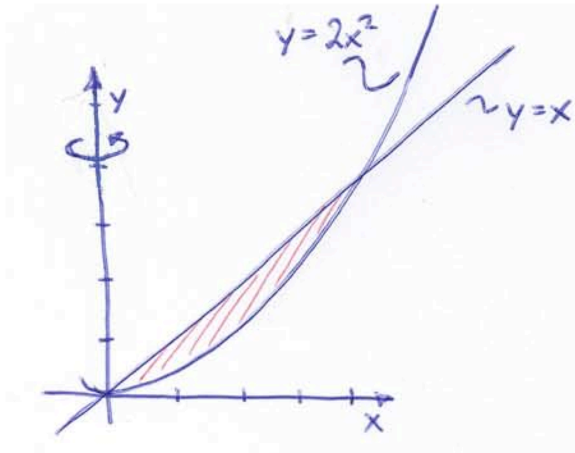
(4 marks) Find the area enclosed by the two functions.



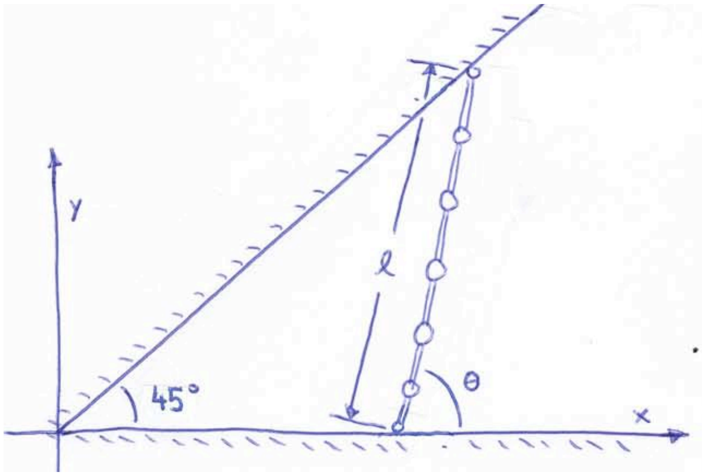
(5 marks) Find the volume between the two functions revolved around the y-axis.

area of a disk = πr^2
area of a shell = $2\pi rh$





(2 marks) BONUS
Maximize the area by translating (moving) and rotating the straight fence section of length L.



(2 marks) Determine the 800th number in the sequence and the sum from the first number to the 800th number for the following series.

-397, -396, -395, -394, …

$a_1 = -397$

$n = 800$

$k = -395 - (-396) = 1$

$= -396 - (-397) =$

so

$a_n = a_1 + (n-1)k$

$= -397 + (800-1)(1)$

$= 402$

$a_n = 402$

and

$S_n = \frac{n}{2}(a_1 + a_n)$

$= \frac{800}{2}(-397 + 402)$

$= 2000$

$S_n = 2000$

(2 marks) If a material degrades to 99.8% of the previous years strength, how strong will the material be (compared to the original strength) after 100 years?

1, 0.998, 0.996004, 0.99401, …

$a_1 = 1$

$n = 100$

$r = \frac{0.996004}{0.998} = \frac{0.998}{1} = 0.998$

so

$a_n = a_1 r^{n-1}$

$= (1)(0.998^{100-1})$

$= 0.8202$

$a_n = 82.0\%$

(2 marks each) Take the derivative with respect to "x" of the following functions.

$$y(x) = 2x^3 + 5sin(2x)$$

$$\frac{dy}{dx} = 6x^2 + 10\cos(2x)$$

$$y(x) = 2(x+1)^3 sin(2x)$$

$$\frac{dy}{dx} = 6(x+1)^2 \sin(2x) + 4(x+1)^3 \cos(2x)$$

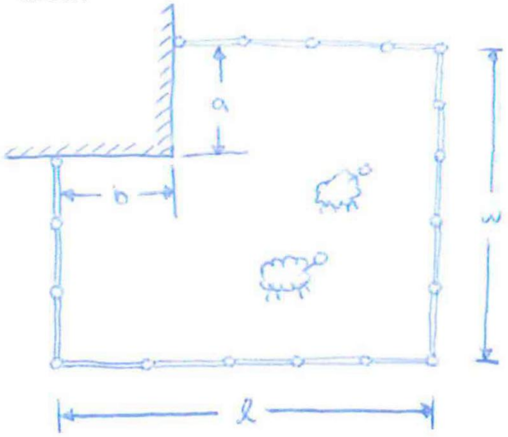
$$y(x) = \lambda x^2 + \alpha \sin(2x) + \rho$$

$$\frac{dy}{dx} = 2\lambda x + 2\alpha \cos(2x)$$

$$V(x) = \frac{(R-r)^2\pi}{3H^2}x^3 + \frac{r\pi}{H}(R-r)x^2 + r^2\pi x$$

$$\frac{dV}{dx} = \frac{(R-r)^2\pi}{H^2}x^2 + 2\frac{r\pi}{H}(R-r)x + r^2\pi$$

(5 marks) You have to build a rectangular fence keeping a corner section that will not be fenced, as shown in the diagram. If you are given "c" units of fencing what dimension would it have to maximize area?



$$2l + 2w - a - b = c \qquad (1)$$

$$lw - ab = A \qquad (2)$$

solve for l in (1)

$$l = \frac{1}{2}(c + a + b - 2w) \qquad (1a)$$

sub (1a) into (2) and take derivative wrt w

$$\left[\frac{1}{2}(c+a+b-2w)\right]w - ab = A$$

$$A = -w^2 + \frac{1}{2}(c+a+b)w - ab$$

$$\frac{dA}{dw} = -2w + \frac{1}{2}(c+a+b) = 0 \qquad \left(\text{find max/min.}\right)$$

$$\boxed{w = \frac{1}{4}(a+b+c)}$$

sub back into (1a)

$$l = \frac{1}{2}\left(c + a + b - 2\left[\frac{1}{4}(a+b+c)\right]\right)$$

$$\boxed{l = \frac{1}{4}(a+b+c)}$$

(2 marks each) Integrate with respect to "x" the following functions.

$$\int 7x^2 + 8\sin(x-1) + \alpha \quad dx$$

$$\frac{7}{3}x^3 - 8\cos(x-1) + \alpha x + c$$

$$\int 2x\sqrt{x^2+1} \quad dx$$

let $u = x^2 + 1$

$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$

so

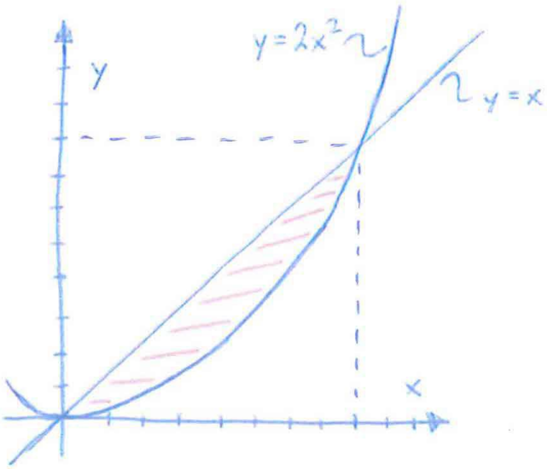
$\int 2x \sqrt{u} \frac{du}{2x}$

$\int u^{\frac{1}{2}} du$

$\frac{2}{3} u^{\frac{3}{2}} + c$

$$\frac{2}{3}(x^2+1)^{\frac{3}{2}} + c$$

(4 marks) Find the area enclosed by the two functions.



Find intercept:

$$y_1 = y_2$$

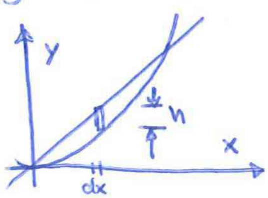
$$2x^2 = x$$

$$2x^2 - x = 0$$

$$x\left(x - \tfrac{1}{2}\right) = 0$$

$$\boxed{x = 0, \tfrac{1}{2}}$$

using columns



$$A = \int h\,dx$$

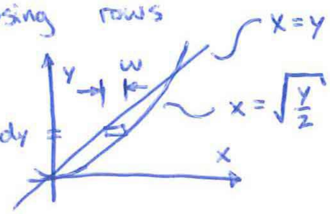
$$= \int_0^{\frac{1}{2}} (x - 2x^2)\,dx$$

$$= \left[\tfrac{1}{2}x^2 - \tfrac{2}{3}x^3\right]_0^{\frac{1}{2}}$$

$$= \left[\tfrac{1}{2}\left(\tfrac{1}{2}\right)^2 - \tfrac{2}{3}\left(\tfrac{1}{2}\right)^3\right] - [0]$$

$$= \frac{1}{24} \cong 0.04167$$

using rows



$$A = \int w\,dy$$

$$= \int_0^{\frac{1}{2}} \left(\frac{y^{\frac{1}{2}}}{\sqrt{2}} - y\right) dy$$

$$= \left[\frac{2}{3\sqrt{2}}\,y^{\frac{3}{2}} - \frac{1}{2}y^2\right]_0^{\frac{1}{2}}$$

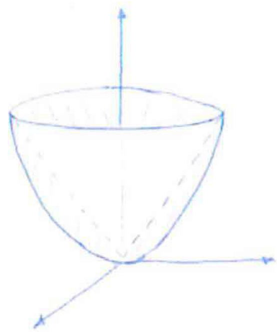
$$= \left[\frac{2}{3\sqrt{2}}\left(\tfrac{1}{2}\right)^{\frac{3}{2}} - \tfrac{1}{2}\left(\tfrac{1}{2}\right)^{2}\right] - [0]$$

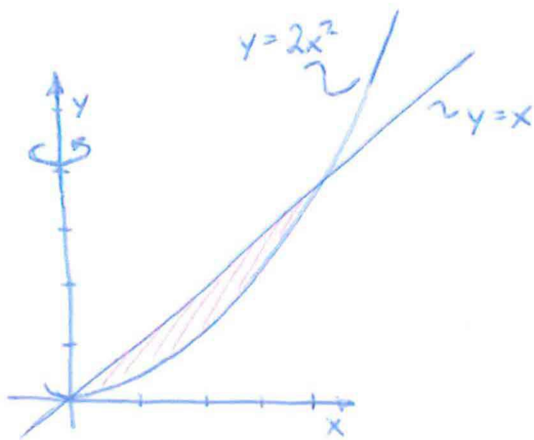
$$= \frac{1}{24} \cong 0.04167$$

$$\boxed{A = \frac{1}{24} \cong 0.04167}$$

(5 marks) Find the volume between the two functions revolved around the y-axis.

area of a disk = πr^2
area of a shell = $2\pi rh$

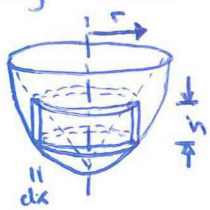




Find intercept:

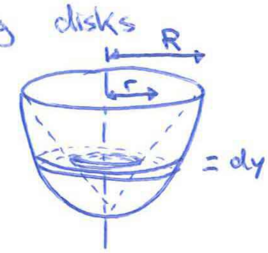
$$y_1 = y_2$$
$$2x^2 = x$$
$$2x^2 - x = 0$$
$$x\left(x - \tfrac{1}{2}\right) = 0 \Rightarrow \boxed{x = 0, \tfrac{1}{2}}$$

using shells



$$V = \int 2\pi r h \, dx$$
$$= \int_0^{\frac{1}{2}} 2\pi [x][x - 2x^2] \, dx$$
$$= \int_0^{\frac{1}{2}} 2\pi x^2 - 4\pi x^3 \, dx$$
$$= \left[\frac{2\pi}{3} x^3 - \pi x^4\right]_0^{\frac{1}{2}}$$
$$= \left[\frac{2\pi}{3}\left(\tfrac{1}{2}\right)^3 - \pi\left(\tfrac{1}{2}\right)^4\right] - [0]$$
$$= \frac{\pi}{48} \cong 0.0654$$

using disks

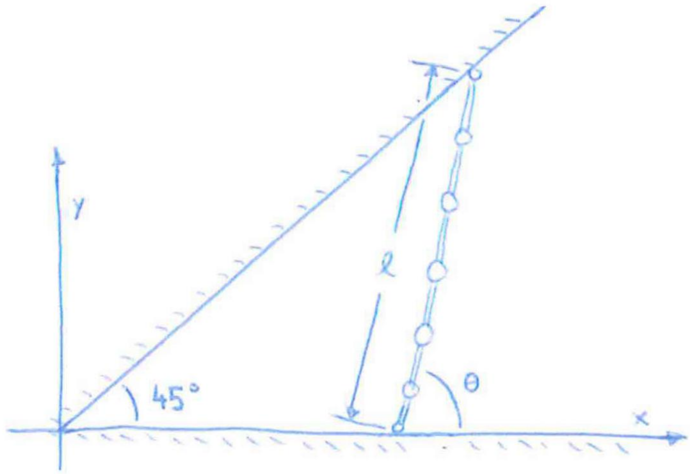


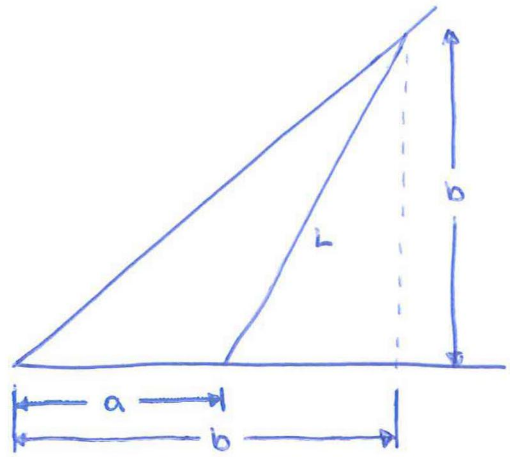
$$V = \int \pi R^2 - \pi r^2 \, dy$$
$$= \int_0^{\frac{1}{2}} \pi\left[\sqrt{\tfrac{y}{2}}\right]^2 - \pi[y]^2 \, dy$$
$$= \int_0^{\frac{1}{2}} \frac{\pi}{2} y - \pi y^2 \, dy$$
$$= \left[\frac{\pi}{4} y^2 - \frac{\pi}{3} y^3\right]_0^{\frac{1}{2}}$$
$$= \left[\frac{\pi}{4}\left(\tfrac{1}{2}\right)^2 - \frac{\pi}{3}\left(\tfrac{1}{2}\right)^3\right] - [0]$$
$$= \frac{\pi}{48} \cong 0.0654$$

$$\boxed{V = \frac{\pi}{48} = 0.0654}$$

(2 marks) BONUS

Maximize the area by translating (moving) and rotating the straight fence section of length L.





$$(b-a)^2 + b^2 = L^2 \qquad (1)$$

$$A = \frac{1}{2}b^2 - \frac{1}{2}b(b-a) \qquad (2)$$

solve for $(b-a)$ in (1)

$$(b-a) = \left(L^2 - b^2\right)^{\frac{1}{2}} \qquad (1a)$$

sub (1a) into (2) and take derivative wrt b

$$A = \frac{1}{2}b^2 - \frac{1}{2}b\left[\left(L^2-b^2\right)^{\frac{1}{2}}\right]$$

$$\frac{dA}{db} = b - \frac{1}{2}\left(L^2-b^2\right)^{\frac{1}{2}} + \frac{1}{2}b^2\left(L^2-b^2\right)^{-\frac{1}{2}} \overset{\text{max/min}}{=} 0$$

$$b\sqrt{L^2-b^2} - \frac{1}{2}\left(L^2-b^2\right) + \frac{1}{2}b^2 = 0$$

$$b\sqrt{L^2-b^2} = \frac{1}{2}L^2 - b^2$$

$$b^2\left(L^2-b^2\right) = \frac{1}{4}L^4 - L^2b^2 + b^4$$

$$2b^4 - 2L^2b^2 + \frac{L^4}{4} = 0$$

$$b^2 = \frac{2L^2 \pm \sqrt{4L^4 - (4)(2)\left(\frac{L^4}{4}\right)}}{(2)(2)}$$

$$b^2 = \left(\frac{1}{2} \pm \frac{\sqrt{2}}{4}\right)L^2$$

so

$$\boxed{b = \sqrt{\frac{1}{2} \pm \frac{\sqrt{2}}{4}}\, L}$$

$$b \cong 0.924L,\ 0.383L$$

sub into (1a)

$$a = b - \left(L^2-b^2\right)^{\frac{1}{2}}$$

⋮

$$a = \sqrt{\frac{1}{2} \pm \frac{\sqrt{2}}{4}}\, L - \sqrt{\frac{1}{2} \mp \frac{\sqrt{2}}{4}}\, L$$

$$\boxed{a = \left[\sqrt{\frac{1}{2} \pm \frac{\sqrt{2}}{4}} - \sqrt{\frac{1}{2} \mp \frac{\sqrt{2}}{4}}\right] L}$$

$$a \cong 0.541L,\ -0.541L$$