Instructor:	Frank Secretain
Course:	Math 20
Assessment:	Test 2
Time allowed:	110 minutes
Devices allowed:	Pencil, pen, eraser, calculator
Notes from instructor:	Be neat. Show your work where needed. Box final answers.
Marks allocated:	6 questions worth 25 marks
Percentage of final grade:	24% of final grade

Formula Sheet

Arithmetic Series

$$a_n = a_1 + (n-1)k$$

$$S_n = \sum_{i=1}^{n} a_1 + (i-1)k$$

$$= \frac{n}{2}(a_1 + a_n)$$

Geometric Series

$$a_n = a_1 r^{n-1}$$

$$S_n = \sum_{i=1}^{n} a_1 r^{i-1}$$

$$= a_1 \frac{1-r^n}{1-r}$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^{n-k} y^k$$

Line equation

$$y = ax + b$$

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition of the derivative

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Rules of differentiation

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$ (product rule)

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$ (quotient rule)

$$\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(f(g(x)))\frac{d}{dx}(g(x))$$ (chain rule)

Derivatives of select functions

$$\frac{d}{dx}(ax^n) = anx^{n-1}$$

$$\frac{d}{dx}(sin(x)) = cos(x)$$

$$\frac{d}{dx}(cos(x)) = -sin(x)$$

$$\frac{d}{dx}(tan(x)) = \frac{1}{(cos(x))^2}$$

$$\frac{d}{dx}(a^x) = a^x ln(a)$$

$$\frac{d}{dx}(log_a x) = \frac{1}{xln(a)}$$

Integrals of select functions

$$\int ax^n dx = \left\{ \begin{array}{ll} \frac{a}{n+1}x^{n+1} & , n \neq -1 \\ aln(|x|) & , n = -1 \end{array} \right\}$$ (polynomials)

$$\int sin(ax)dx = -\frac{1}{a}cos(ax)$$

$$\int cos(ax)dx = \frac{1}{a}sin(ax)$$ (trigonometry)

$$\int tan(ax)dx = \frac{1}{a}ln(|sec(ax)|)$$

$$\int a^x dx = \frac{1}{ln(a)}a^x$$

$$\int ln(x)dx = xln(x) - x$$ (exponentials)

Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n$$

Integration by parts

$$\int udv = uv - \int vdu$$

Integrate (3 marks)

$$\int 7x^2 + 8\sin(x-1) + \alpha \quad dx$$

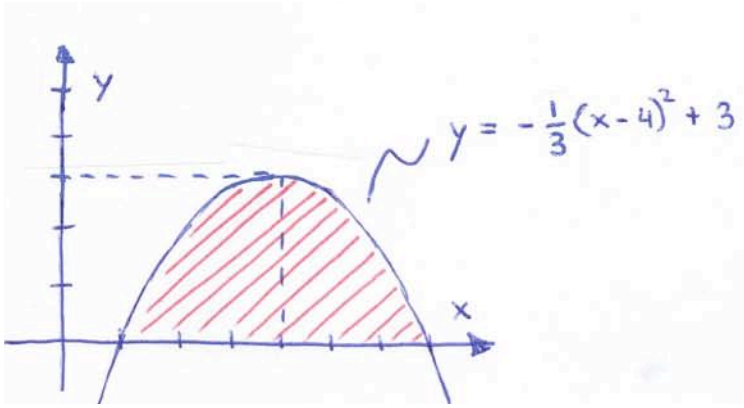
Integrate (3 marks)

$$\int 2x\sqrt{x^2+1} \;\; dx$$

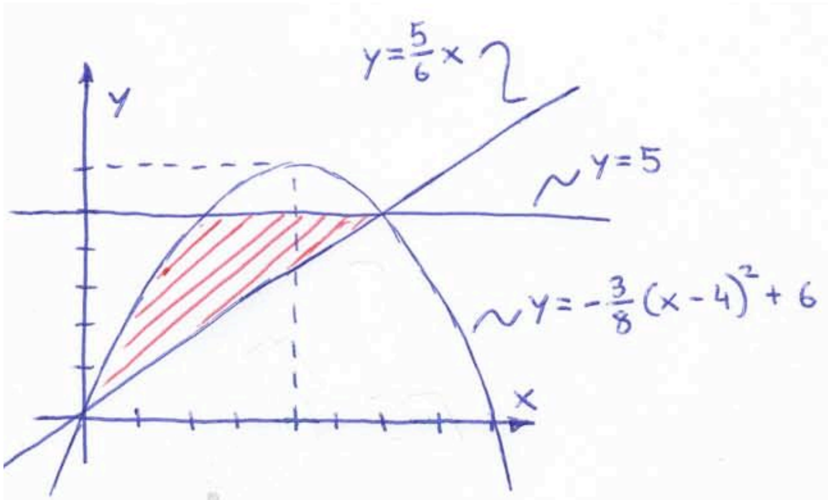
Integrate (4 marks)

$$\int 2x\sqrt{x+1} + 3x \quad dx$$

Find the area of the shaded section (5 marks)



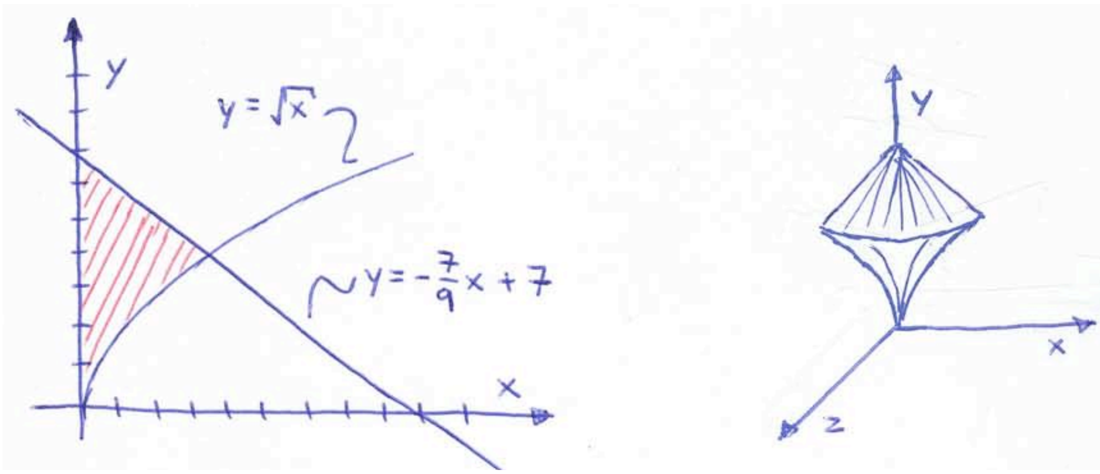
Find the area of the shaded section (5 marks)



Find the volume between the y-axis and the two curves revolved around the y-axis (5 marks).
Remember that:

area of a disk = πr^2

area of a shell = $2\pi rh$



Integrate (3 marks)

$$\int 7x^2 + 8\sin(x-1) + \alpha \;\; dx$$

$$\boxed{\frac{7}{3}x^3 - 8\cos(x-1) + \alpha x + c}$$

Integrate (3 marks)

$$\int 2x\sqrt{x^2+1}\ \ dx$$

let $u = x^2+1$

$\frac{du}{dx} = 2x \quad \Rightarrow \quad dx = \frac{du}{2x}$

substitute:

$\int \cancel{2x}\sqrt{u}\,\frac{du}{\cancel{2x}}$

$\int u^{\frac{1}{2}}\,du$

$\frac{2}{3}u^{\frac{3}{2}} + c$

$\boxed{\frac{2}{3}(x^2+1)^{\frac{3}{2}} + c}$

Integrate (4 marks)

$$\int 2x\sqrt{x+1} + 3x \quad dx$$

$$\int 2x\sqrt{x+1}\, dx \qquad + \int 3x\, dx$$

let $u = 2x$ $\quad \int dv = \int (x+1)^{\frac{1}{2}}\, dx$

$du = 2dx$ $\quad v = \frac{2}{3}(x+1)^{\frac{3}{2}}$

by parts

$$\int u\,dv = uv - \int v\,du$$

$$\int 2x\sqrt{x+1}\, dx = (2x)\left(\frac{2}{3}(x+1)^{\frac{3}{2}}\right) - \int \frac{2}{3}(x+1)^{\frac{3}{2}}\, 2dx$$

$$= \frac{4x}{3}(x+1)^{\frac{3}{2}} - \frac{4}{3}\frac{2}{5}(x+1)^{\frac{5}{2}}$$

and

$$\int 3x\, dx = \frac{3}{2}x^2$$

so

$$\boxed{\int 2x\sqrt{x+1}\, dx = \frac{4x}{3}(x+1)^{\frac{3}{2}} - \frac{8}{15}(x+1)^{\frac{5}{2}} + \frac{3}{2}x^2 + C}$$

Integrate (4 marks)

$$\int 2x\sqrt{x+1} + 3x \quad dx$$

$$\int 2x\sqrt{x+1}\, dx \quad + \quad \int 3x\, dx$$

let $u = x+1 \quad \Rightarrow \quad x = u-1$

$\frac{du}{dx} = 1 \quad \Rightarrow \quad dx = du$

substitute:

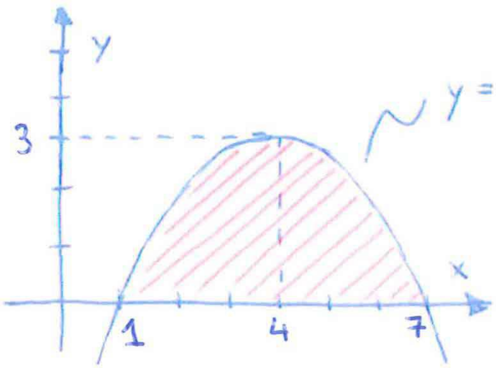
$$\int 2(u-1)(u)^{\frac{1}{2}}\, du \quad + \quad \int 3x\, dx$$

$$\int 2u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\, du \quad + \quad \frac{3}{2}x^2$$

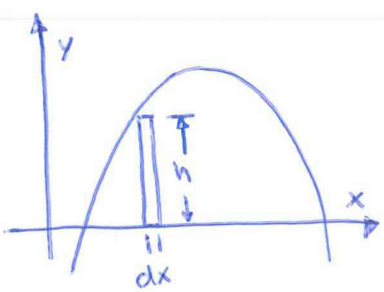
$$\frac{4}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}} + \frac{3}{2}x^2 + c$$

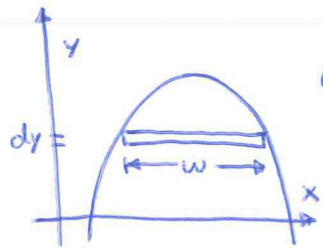
$$\boxed{\frac{4}{5}(x+1)^{\frac{5}{2}} - \frac{4}{3}(x+1)^{\frac{3}{2}} + \frac{3}{2}x^2 + c}$$

Find the area of the shaded section (5 marks)



$y = -\frac{1}{3}(x-4)^2 + 3$





$y = -\frac{1}{3}(x-4)^2 + 3$

$(x-4)^2 = 9 - 3y$

$x = \pm\sqrt{9-3y} + 4$

$dA = h\,dx$

$\int dA = \int \left(-\frac{1}{3}(x-4)^2 + 3\right) dx$

$A = \int_1^7 -\frac{1}{3}(x-4)^2 + 3\,dx$

$= \left[-\frac{1}{9}(x-4)^3 + 3x\right]_1^7$

$= \left[-\frac{1}{9}(7-4)^3 + 3(7)\right] - \left[-\frac{1}{9}(1-4)^3 + 3(1)\right]$

$= -3 + 21 - 3 - 3$

$\boxed{A = 12}$

$dA = w\,dy$

$= \left(\left[\sqrt{9-3y} + 4\right] - \left[-\sqrt{9-3y} + 4\right]\right) dy$

$\int dA = \int 2\sqrt{9-3y}\,dy$

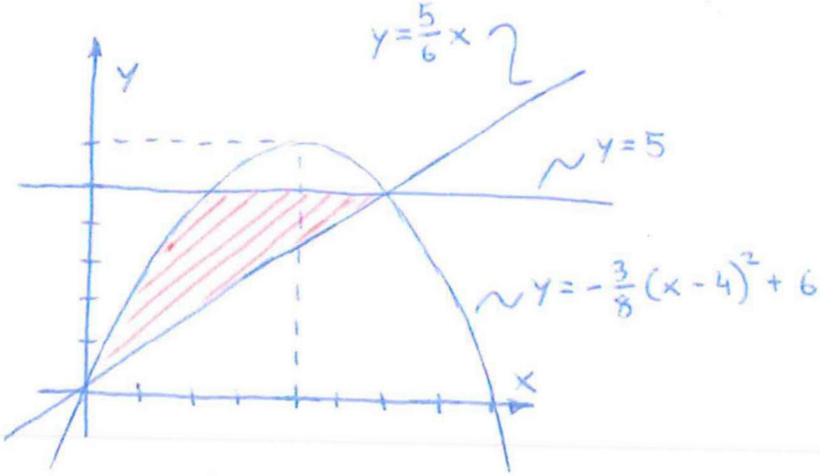
$A = \int_0^3 2(9-3y)^{\frac{1}{2}}\,dy$

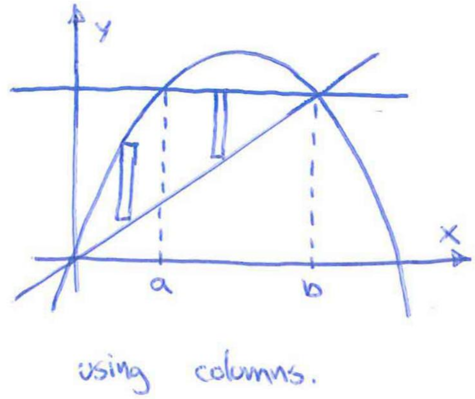
$= \left[\frac{4}{3}(9-3y)^{\frac{3}{2}}\left(-\frac{1}{3}\right)\right]_0^3$

$= \left[-\frac{4}{9}\cancel{(9-3(3))^{\frac{3}{2}}}^{\,0}\right] - \left[-\frac{4}{9}(9-3(0))^{\frac{3}{2}}\right]$

$\boxed{A = 12}$

Find the area of the shaded section (5 marks)





using columns.

$$A=\int_0^a\left(\left[-\frac{3}{8}(x-4)^2+6\right]-\left[\frac{5}{6}x\right]\right)dx+\int_a^b\left([5]-\left[\frac{5}{6}x\right]\right)dx$$

$$=\left[-\frac{1}{8}(x-4)^3-\frac{5}{12}x^2+6x\right]_0^a+\left[-\frac{5}{12}x^2+5x\right]_a^b$$

see additional page for intercepts, $a=2.367$ & $b=\begin{Bmatrix}5.633\\5.778\\6\end{Bmatrix}$

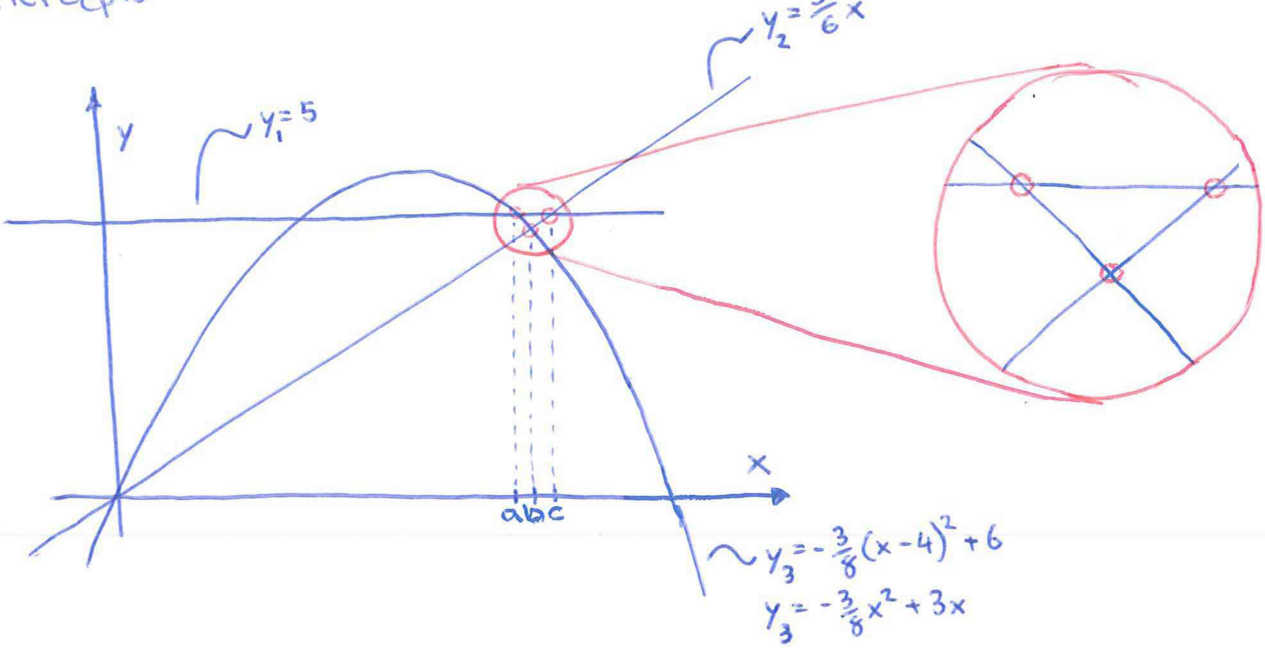
$$A=\left[-\frac{1}{8}(2.367-4)^3-\frac{5}{12}(2.367)^2+6(2.367)\right]-\left[-\frac{1}{8}(0-4)^3-\cancel{\frac{5}{12}(0)^2}+\cancel{6(0)}\right]+$$

$$\left[-\frac{5}{12}b^2+5b\right]-\left[-\frac{5}{12}(2.367)^2+5(2.367)\right]$$

$$=-\frac{5}{12}b^2+5b-5.089$$

$$\boxed{\begin{aligned}A(b=5.633)&=9.85\\A(b=5.778)&=9.89\\A(b=6)&=9.91\end{aligned}}$$

Intercepts:



point a:

$$y_1 = y_3$$

$$5 = -\frac{3}{8}x^2 + 3x$$

$$\frac{3}{8}x^2 - 3x + 5 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= 4 \pm \frac{4}{3}\sqrt{\frac{3}{2}}$$

$$= 5.633 \,,\ 2.367$$

$$\boxed{a = 5.633}$$

point b:

$$y_2 = y_3$$

$$\frac{5}{6}x = -\frac{3}{8}x^2 + 3x$$

$$(x)\left(-\frac{3}{8}x + \frac{13}{6}\right) = 0$$

$$x = 0 \quad \Rightarrow x = 0$$

$$-\frac{3}{8}x + \frac{13}{6} = 0 \quad \Rightarrow x = \frac{52}{9}$$

$$\boxed{b = \frac{52}{9} \cong 5.778}$$

point c:

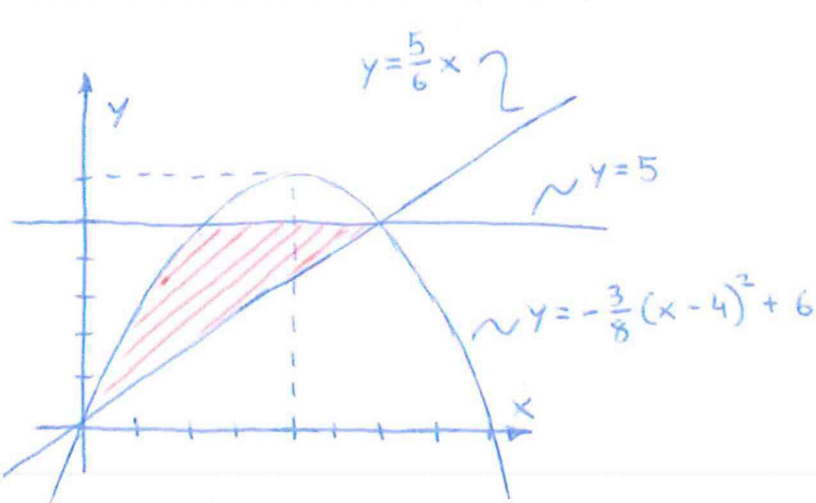
$$y_1 = y_2$$

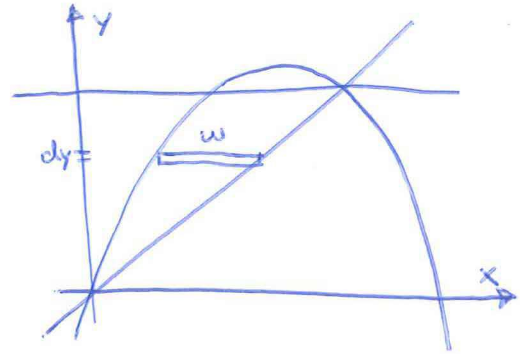
$$5 = \frac{5}{6}x$$

$$x = 6$$

$$\boxed{c = 6}$$

Find the area of the shaded section (5 marks)





using rows

$$y=-\frac{3}{8}(x-4)^2+6 \qquad y=\frac{5}{6}x$$

$$(x-4)^2=\frac{48}{3}-\frac{8}{3}y \qquad x=\frac{6}{5}y$$

$$x=\pm\sqrt{\frac{48}{3}-\frac{8}{3}y}+4$$

so

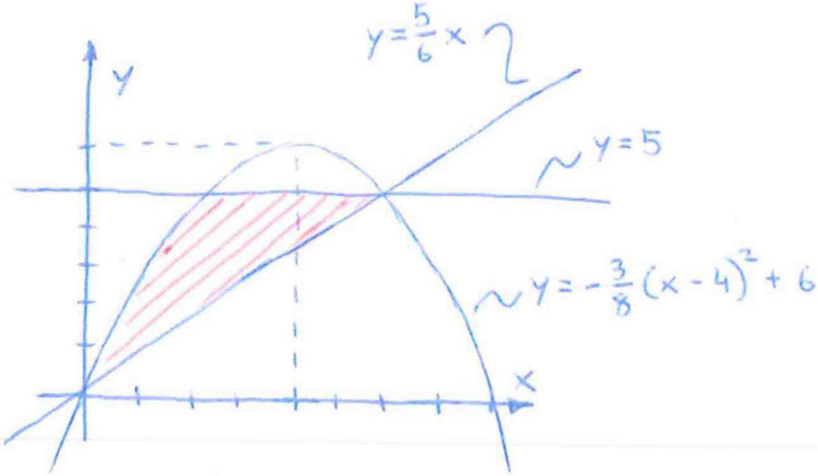
$$A=\int_0^5\left(\left[\frac{6}{5}y\right]-\left[-\sqrt{\frac{48}{3}-\frac{8}{3}y}+4\right]\right)dy$$

$$=\left[\frac{6}{10}y^2-\frac{1}{4}\left(\frac{48}{3}-\frac{8}{3}y\right)^{\frac{3}{2}}-4y\right]_0^5$$

$$=\left[\frac{6}{10}(5)^2-\frac{1}{4}\left(\frac{48}{3}-\frac{8}{3}(5)\right)^{\frac{3}{2}}-4(5)\right]-\left[\cancel{\frac{6}{10}(0)^2}^{0}-\frac{1}{4}\left(\frac{48}{3}-\frac{8}{3}(0)\right)^{\frac{3}{2}}-\cancel{4(0)}^{0}\right]$$

$$\boxed{A=9.91}$$

Find the area of the shaded section (5 marks)



using columns:

$$A = \int_0^{2.367} \left[\left(-\tfrac{3}{8}(x-4)^2+6\right) - \left(\tfrac{5}{6}x\right)\right] dx + \int_{2.367}^{5.633} \left[(5) - \left(\tfrac{5}{6}x\right)\right] dx + \int_{5.633}^{5.778} \left[\left(-\tfrac{3}{8}(x-4)^2+6\right) - \left(\tfrac{5}{6}x\right)\right] dx$$

$$= \left[-\tfrac{1}{8}(x-4)^3 - \tfrac{5}{12}x^2 + 6x\right]_0^{2.367} + \left[-\tfrac{5}{12}x^2 + 5x\right]_{2.367}^{5.633} + \left[-\tfrac{1}{8}(x-4)^3 - \tfrac{5}{12}x^2 + 6x\right]_{5.633}^{5.778}$$

$$\boxed{A = 9.87}$$

using rows

intercepts: $y(x = 5.778) = 4.815$

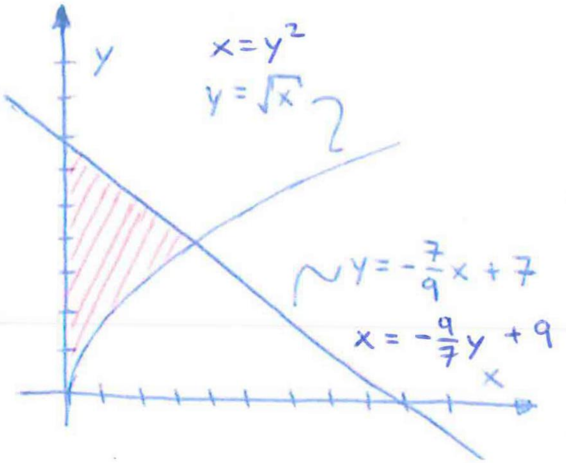
$$A = \int_0^{4.815} \left[\left(\tfrac{6}{5}y\right) - \left(-\sqrt{\tfrac{48}{3} - \tfrac{8}{3}y} + 4\right)\right] dy + \int_{4.815}^{5} \left[\left(+\sqrt{\tfrac{48}{3} - \tfrac{8}{3}y} + 4\right) - \left(-\sqrt{\tfrac{48}{3} - \tfrac{8}{3}y} + 4\right)\right] dy$$

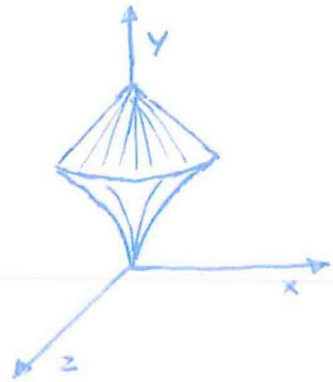
$$= \left[\tfrac{6}{10}y^2 - \tfrac{1}{4}\left(\tfrac{48}{3} - \tfrac{8}{3}y\right)^{\frac{3}{2}} - 4y\right]_0^{4.815} + \left[-\tfrac{1}{2}\left(\tfrac{48}{3} - \tfrac{8}{3}y\right)^{\frac{3}{2}}\right]_{4.815}^{5}$$

$$\boxed{A = 9.87}$$

Find the volume between the y-axis and the two curves revolved around the y-axis (5 marks).
Remember that:

area of a disk = πr^2
area of a shell = $2\pi rh$





intercept:

$$y_1 = y_2$$

$$(\sqrt{x})^2 = \left(-\frac{7}{9}x + 7\right)^2$$

$$x = \frac{49}{81}x^2 - \frac{98}{9}x + 49$$

$$\frac{49}{81}x^2 - \frac{107}{9} + 49 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{963}{98} \pm \frac{81}{98}\sqrt{\frac{205}{9}}$$

$$= 13.77 \,,\, 5.88$$

$y(x = 5.88) = 2.43$

disks

$$V = \int_0^{2.43} \pi (y^2)^2 \, dy + \int_{2.43}^{7} \pi \left(-\frac{9}{7}y + 9\right)^2 dy$$

$$= \left[\frac{\pi}{5}y^5\right]_0^{2.43} + \left[-\frac{7\pi}{27}\left(-\frac{9}{7}y + 9\right)^3\right]_{2.43}^{7} = 218$$

shells

$$V = \int_0^{5.88} 2\pi x \left[\left(-\frac{7}{9}x + 7\right) - \sqrt{x}\right] dx = 2\pi \int_0^{5.88} -\frac{7}{9}x^2 - x^{\frac{3}{2}} + 7x \, dx$$

$$= 2\pi \left[-\frac{7}{27}x^3 - \frac{2}{5}x^{\frac{5}{2}} + \frac{7}{2}x^2\right]_0^{5.88} = 218$$