Instructor:	Frank Secretain
Course:	Math 20
Assessment:	Final
Time allowed:	110 minutes
Devices allowed:	Pencil, pen, eraser, calculator.
Notes from instructor:	Be neat. Show your work where needed. Box final answers.
Marks allocated:	7 questions worth 25 marks
Percentage of final grade:	25% of final grade

Formula Sheet

Arithmetic Series

$$a_n = a_1 + (n-1)k$$

$$S_n = \sum_{i=1}^{n} a_1 + (i-1)k$$

$$= \frac{n}{2}(a_1 + a_n)$$

Geometric Series

$$a_n = a_1 r^{n-1}$$

$$S_n = \sum_{i=1}^{n} a_1 r^{i-1}$$

$$= a_1 \frac{1-r^n}{1-r}$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^{n-k} y^k$$

Line equation

$$y = ax + b$$

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition of the derivative

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Rules of differentiation

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$ (product rule)

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$ (quotient rule)

$$\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(f(g(x)))\frac{d}{dx}(g(x))$$ (chain rule)

Derivatives of select functions

$$\frac{d}{dx}(ax^n) = anx^{n-1}$$

$$\frac{d}{dx}(sin(x)) = cos(x)$$

$$\frac{d}{dx}(cos(x)) = -sin(x)$$

$$\frac{d}{dx}(tan(x)) = \frac{1}{(cos(x))^2}$$

$$\frac{d}{dx}(a^x) = a^x ln(a)$$

$$\frac{d}{dx}(log_a x) = \frac{1}{xln(a)}$$

Integrals of select functions

$$\int ax^n dx = \begin{cases} \frac{a}{n+1}x^{n+1} & , n \neq -1 \\ aln(|x|) & , n = -1 \end{cases}$$ (polynomials)

$$\int sin(ax)dx = -\frac{1}{a}cos(ax)$$

$$\int cos(ax)dx = \frac{1}{a}sin(ax)$$ (trigonometry)

$$\int tan(ax)dx = \frac{1}{a}ln(|sec(ax)|)$$

$$\int a^x dx = \frac{1}{ln(a)}a^x$$

$$\int ln(x)dx = xln(x) - x$$ (exponentials)

Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n$$

Integration by parts

$$\int udv = uv - \int vdu$$

(2 marks) Determine the 300th number in the sequence and the sum from the first number to the 300th number.

100, -99, 98.01, …

(2 marks) It can be argued that after driving 1 million kilometres you are concerned an expert driver. Suppose, on your first year of driving you drive 5000 km and for every year after you drive 1000 km more than the previous year (i.e. on the second year you drive 6000 km and the third 7000 km). How many years of driving must you drive to achieve 1 million kilometres.

(2 marks) Take the derivative with respect to “x” of the following function **using the definition of the derivative**.

$$y = 2x^2 + c$$

(5 marks) Take the derivative with respect to “x” of the following functions:

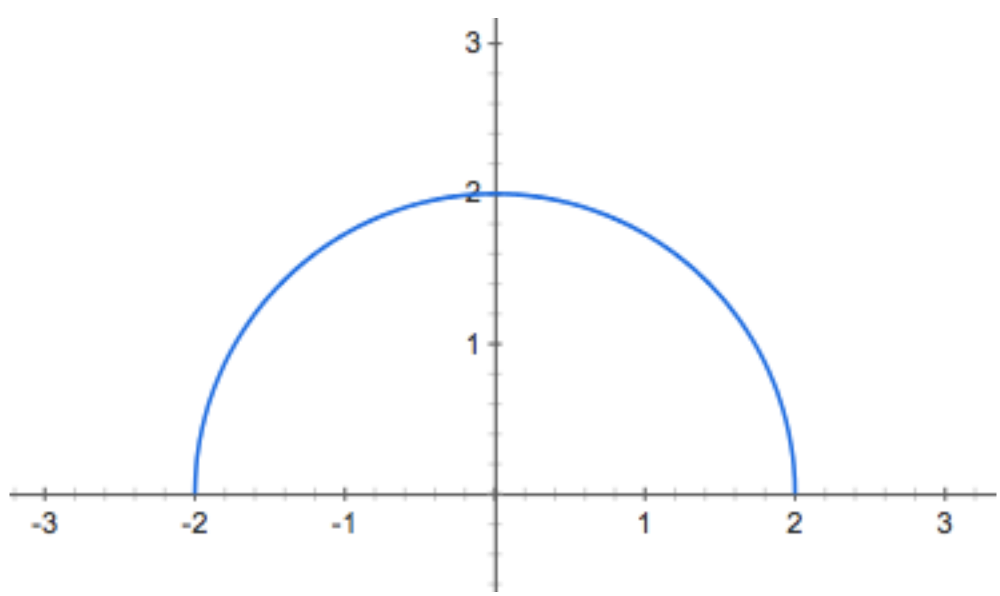
$$y = 2x^2 + c$$

$$y = 5x^3 \sin(x) + 8x$$

$$y = 4x + \cos(1 + (x^2 - 1)^3)$$

(5 marks) Determine the tangent line at x=-1 for the following function. Plot the tangent line on the plot.

$$y = \sqrt{2^2 - x^2}$$

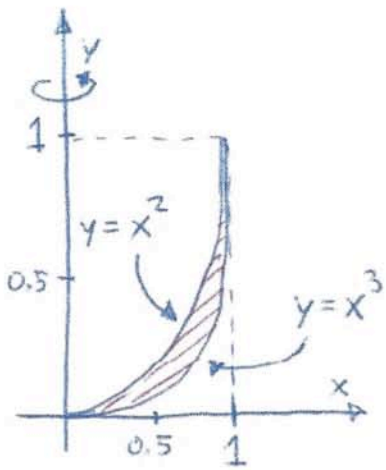


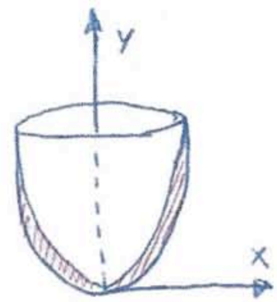
(4 marks) Integrate the following functions:

$$\int 4x^{\frac{1}{2}} + \sin(3x)\ dx$$

$$\int ax\cos(3x) + b\ dx$$

(5 marks) Find the volume between the two curves revolved around the y-axis.





(2 marks) Determine the 300th number in the sequence and the sum from the first number to the 300th number.

100, -99, 98.01, …

$k = (-99) - (100) = -199$
$= (98.01) - (-99) = 197.01$ ×

$r = \frac{-99}{100} = -0.99$ ✓ (geometric)
$= \frac{98.01}{-99} = -0.99$

$a_n = a_1 r^{n-1}$
$= (100)(-0.99)^{300-1}$

$a_n = -4.954$

$S_n = a_n \frac{1-r^n}{1-r}$
$= (100)\frac{1-(-0.99)^{300}}{1-(-0.99)}$

$S_n = 47.79$

(2 marks) It can be argued that after driving 1 million kilometres you are concerned an expert driver. Suppose, on your first year of driving you drive 5000 km and for every year after you drive 1000 km more than the previous year (i.e. on the second year you drive 6000 km and the third 7000 km). How many years of driving must you drive to achieve 1 million kilometres.

5000, 6000, 7000, …

$k = 6000 - 5000 = 1000$ ✓ (arithmetic)
$= 7000 - 6000 = 1000$

$r = \frac{6000}{5000} = 1.2$ ×
$= \frac{7000}{6000} = 1.1\overline{6}$

$a_1 = 5000$
$a_n = ?$
$k = 1000$
$S_n = 1\,000\,000$
$n = ?$

$a_n = a_1 + (n-1)k$ (1)
$S_n = \frac{n}{2}(a_1 + a_n)$ (2)

sub (1) into (2)

$S_n = \frac{n}{2}(a_1 + [a_1 + (n-1)k])$
$2S_n = 2a_1 n + n^2 k - nk$
$0 = kn^2 + (2a_1 - k)n - 2S_n$
$0 = 1000n^2 + (2(5000) - 1000)n - 2(1000000)$
$0 = 1000n^2 + 9000n - 2000000$
$0 = n^2 + 9n - 2000$

$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$= \frac{-9 \pm \sqrt{81 - 4(-2000)}}{2(1)}$
$= -4.5 \pm 44.95$
$= 40.45, -49.45$

$n = 41$ years

(2 marks) Take the derivative with respect to "x" of the following function **using the definition of the derivative**.

$$y = 2x^2 + c$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[2(x+\Delta x)^2 + c\right] - \left[2x^2 + c\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x^2 + 4x\Delta x + 2\Delta x^2 + c - 2x^2 - c}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{4x\Delta x + 2\Delta x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 4x + 2\Delta x$$

$$= 4x$$

$$\boxed{\frac{dy}{dx} = 4x}$$

(5 marks) Take the derivative with respect to "x" of the following functions:

$$y = 2x^2 + c$$

$$\boxed{\frac{dy}{dx} = 4x}$$

$$y = 5x^3 \sin(x) + 8x$$

$$\frac{dy}{dx} = 15x^2 \sin(x) + 5x^3 \cos(x) + 8$$

$$y = 4x + \cos(1 + (x^2 - 1)^3)$$

$$\frac{dy}{dx} = 4 - \sin\left(1 + (x^2-1)^3\right)\left(3(x^2-1)^2\right)\left(2x\right)$$

(5 marks) Determine the tangent line at x=-1 for the following function. Plot the tangent line on the plot.

$$y = \sqrt{2^2 - x^2}$$

$$y = (4 - x^2)^{\frac{1}{2}}$$

$$y(-1) = (4 - [-1]^2)^{\frac{1}{2}}$$
$$= 1.73205$$

$$\frac{dy}{dx} = \frac{1}{2}(4 - x^2)^{-\frac{1}{2}}(-2x)$$

$$\left.\frac{dy}{dx}\right|_{x=-1} = \frac{1}{2}(4 - [-1]^2)^{-\frac{1}{2}}(-2[-1])$$
$$= 0.57735$$
$$= \text{slope} = a$$

$$y = ax + b$$

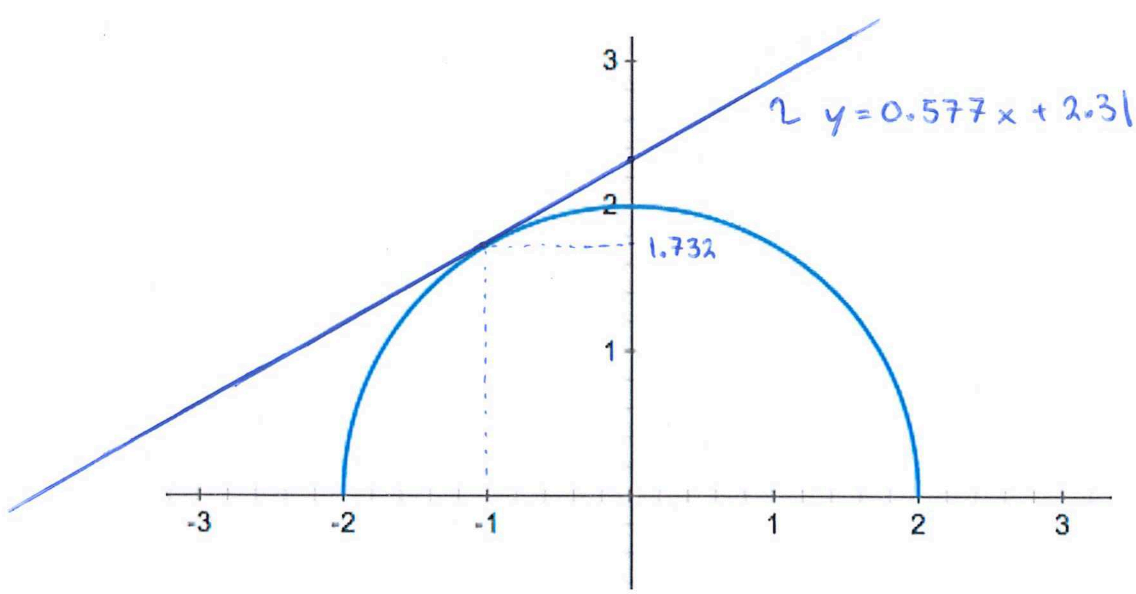
$$y = (0.57735)x + b$$

sub in $(x, y) = (-1, 1.73205)$

$$1.73205 = (0.57735)(-1) + b$$

$$b = 2.3094$$

$$\boxed{y = 0.577x + 2.31}$$



(4 marks) Integrate the following functions:

$$= \int 4x^{\frac{1}{2}} + \sin(3x)\ dx$$

$$= 4 \cdot \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{3}\cos(3x) + c$$

$$= \frac{8}{3} x^{\frac{3}{2}} - \frac{1}{3}\cos(3x) + c$$

$$= \int ax\cos(3x) + b\ dx$$

$$= \int ax\cos(3x)\ dx + \int b\ dx$$

let $u = ax$ $\quad \int dv = \int \cos(3x)\ dx$

$\frac{du}{dx} = a$ $\quad V = \frac{1}{3}\sin(3x)$

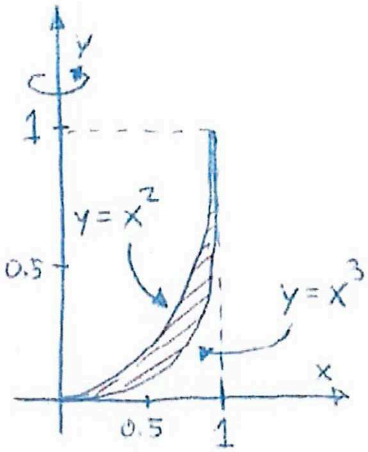
$du = a\,dx$

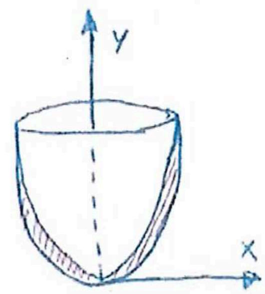
$$= (ax)\left(\frac{1}{3}\sin(3x)\right) - \int\left(\frac{1}{3}\sin(3x)\right)(a\,dx) + bx$$

$$= \frac{ax}{3}\sin(3x) - \frac{a}{3}\left(-\frac{1}{3}\cos(3x)\right) + bx + c$$

$$= \frac{ax}{3}\sin(3x) + \frac{a}{9}\cos(3x) + bx + c$$

(5 marks) Find the volume between the two curves revolved around the y-axis.



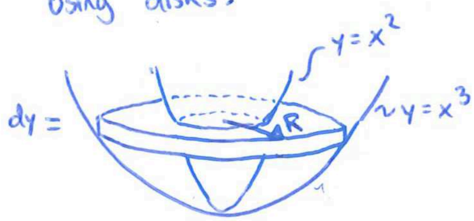


solve for x:

$y = x^2 \qquad y = x^3$

$x = \pm\sqrt{y} \qquad x = \sqrt[3]{x}$

using disks:



$$d\forall = \left(\pi R^2 - \pi r^2\right) dy$$

$$= \pi\left(\sqrt[3]{y}\right)^2 - \pi\left(\sqrt[2]{y}\right)^2 dy$$

$$\int d\forall = \int_0^1 \pi y^{\frac{2}{3}} - \pi y \ dy$$

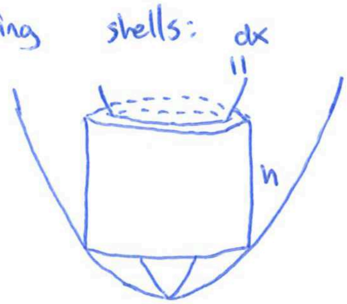
$$\forall = \left[\frac{3\pi}{5} y^{\frac{5}{3}} - \frac{\pi}{2} y^2\right]_0^1$$

$$= \left[\frac{3\pi}{5}(1)^{\frac{5}{3}} - \frac{\pi}{2}(1)^2\right] - [0]$$

$$= \frac{3\pi}{5} - \frac{\pi}{2} = \frac{6\pi}{10} - \frac{5\pi}{10}$$

$$\boxed{\forall = \frac{\pi}{10} \cong 0.314}$$

using shells:



$$d\forall = 2\pi R h \ dx$$

$$= 2\pi x \left(y_1 - y_2\right) dx$$

$$= 2\pi x \left(x^2 - x^3\right) dx$$

$$\int d\forall = \int_0^1 2\pi x^3 - 2\pi x^4 \ dx$$

$$\forall = \left[\frac{2\pi}{4} x^4 - \frac{2}{5}\pi x^5\right]_0^1$$

$$= \left[\frac{2\pi}{4}(1)^4 - \frac{2}{5}\pi(1)^5\right] - [0]$$

$$= \frac{2\pi}{4} - \frac{2\pi}{5} = \frac{10\pi}{20} - \frac{8\pi}{20}$$

$$\boxed{\forall = \frac{2\pi}{20} = \frac{\pi}{10} \cong 0.314}$$