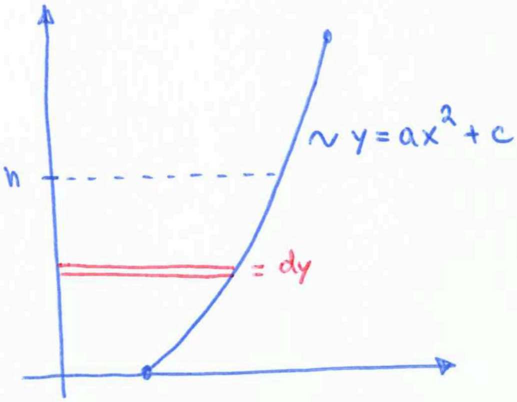
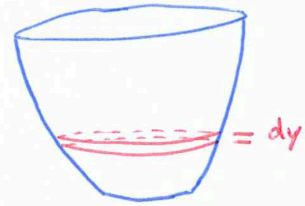
Truncated Parabolic Bowl Volume (using disks)





$$d\forall = \pi r^2 dy$$

using disks

$$\int d\forall = \int_0^h \pi x^2 dy = \int_0^h \pi \left[\frac{y-c}{a}\right] dy$$

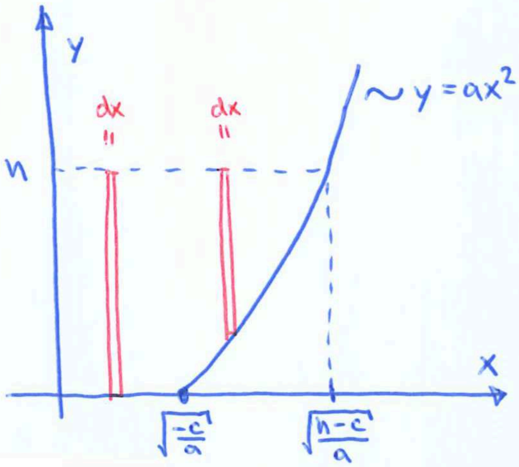
$$\forall = \left[\frac{\pi a}{2}\left(\frac{y-c}{a}\right)^2\right]_0^h = \frac{\pi a}{2}\left(\frac{h-c}{a}\right)^2 - \frac{\pi a}{2}\left(\frac{0-c}{a}\right)^2$$

$$= \frac{\pi}{2a}\left((h-c)^2 - c^2\right) = \frac{\pi}{2a}\left(h^2 - 2hc + \cancel{c^2} - \cancel{c^2}\right)$$

$$= \frac{\pi h}{2a}\left(h - 2c\right)$$

$$\boxed{\forall(h) = \frac{\pi h}{2a}\left(h-2c\right)}$$

Truncated Parabolic Bowl Volume (using shells)



$$d\forall = 2\pi r h\, dx$$

Using shells

$$\int d\forall = \int_0^{\sqrt{\frac{-c}{a}}} 2\pi x h\, dx + \int_{\sqrt{\frac{-c}{a}}}^{\sqrt{\frac{h-c}{a}}} 2\pi x (h-y)\, dx$$

$$= \left[\pi h x^2\right]_0^{\sqrt{\frac{-c}{a}}} + \int_{\sqrt{\frac{-c}{a}}}^{\sqrt{\frac{h-c}{a}}} 2\pi x h - 2\pi x\left[ax^2 + c\right] dx$$

$$= \left[\pi h\left(\frac{-c}{a}\right) - 0\right] + \left[\pi h x^2 - \frac{2}{4}\pi a x^4 - \pi c x^2\right]_{\sqrt{\frac{-c}{a}}}^{\sqrt{\frac{h-c}{a}}}$$

$$= \frac{-\pi h c}{a} + \left[\pi h\left(\frac{h-c}{a}\right) - \frac{\pi a}{2}\left(\frac{h-c}{a}\right)^2 - \pi c\left(\frac{h-c}{a}\right)\right] - \left[\pi h\left(\frac{-c}{a}\right) - \frac{\pi a}{2}\left(\frac{-c}{a}\right)^2 - \pi c\left(\frac{-c}{a}\right)\right]$$

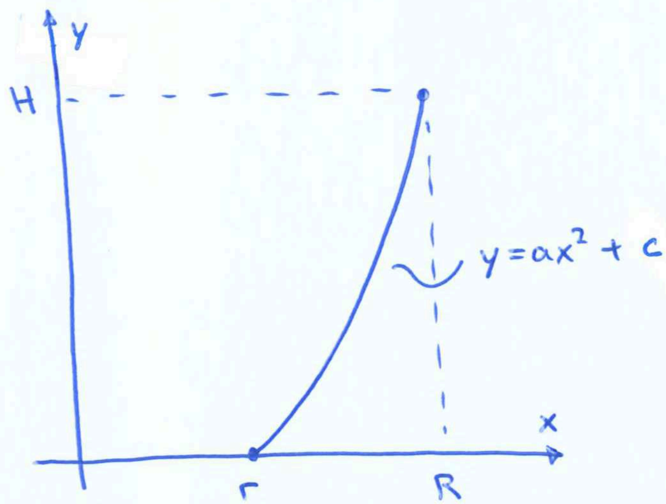
$$= \frac{\pi h}{a}\left[-c + (h-c) - \frac{1}{2h}\left(h^2 - 2hc + c^2\right) - \frac{c}{h}(h-c) + c + \frac{c^2}{2h} - \frac{c^2}{h}\right]$$

$$= \frac{\pi h}{a}\left[-c + h \cancel{-c} - \frac{h}{2} \cancel{+c} \cancel{-\frac{c^2}{2h}} \cancel{-c} + \cancel{\frac{c^2}{h}} \cancel{+c} + \cancel{\frac{c^2}{2h}} - \cancel{\frac{c^2}{h}}\right]$$

$$= \frac{\pi h}{2a}\left[h - 2c\right]$$

$$\boxed{\forall(h) = \frac{\pi h}{2a}(h - 2c)}$$

Truncated Parabolic Bowl Parameters



substitute $x = r$, $y = 0$: $[0] = a[r]^2 + c$ (1)

$x = R$, $y = H$: $[H] = a[R]^2 + c$ (2)

solve (1) for c:

$$c = -ar^2 \tag{1a}$$

sub. (1a) into (2)

$$H = aR^2 + [-ar^2]$$

$$\boxed{a = \frac{H}{R^2 - r^2}} \tag{2a}$$

sub (2a) into (1a):

$$c = -\left[\frac{H}{R^2 - r^2}\right] r^2$$

$$\boxed{c = -\frac{Hr^2}{R^2 - r^2}}$$

Truncated Parabolic Bowl Change in volume

$$\forall(h) = \frac{\pi h}{2a}\left(h - 2c\right) = \frac{\pi}{2a}\left(h^2 - 2ch\right)$$

$\frac{d}{dt}$

$$\frac{d\forall}{dt} = \frac{\pi}{2a}\left(2h\frac{dh}{dt} - 2c\frac{dh}{dt}\right)$$

$$= \frac{\pi}{a}\frac{dh}{dt}\left(h - c\right)$$

$$\boxed{dh = \frac{a}{\pi(h-c)}\, d\forall}$$