# Feynman Integration

$$\int_{-\infty}^{+\infty} \frac{\sin(x)}{x}\, dx$$

$$f(x) = \frac{\sin(x)}{x}$$

$$f(-x) = \frac{\sin(-x)}{-x} = \frac{-\sin(x)}{-x} = \frac{\sin(x)}{x} = f(x) \quad \therefore \text{ even function}$$

$$\int_{-\infty}^{+\infty} \frac{\sin(x)}{x}\, dx = 2\int_0^{\infty} \frac{\sin(x)}{x}\, dx$$

## Feynman Integration

let

$$I(a) = \int_0^{\infty} \frac{\sin(ax)}{x}\, dx$$

so

$$\frac{d}{da} I(a) = \frac{d}{da}\int_0^{\infty} \frac{\sin(ax)}{x}\, dx = \int_0^{\infty} \frac{\partial}{\partial a} \frac{\sin(ax)}{x}\, dx = \int_0^{\infty} \cos(ax)\, dx$$

doesn't work

$-1 \leq \cos(ax) \leq +1$

let

$$I(a) = \int_0^{\infty} \frac{\sin x}{x} e^{-ax}\, dx$$

so

$$\frac{d}{da} I(a) = \frac{d}{da}\int_0^{\infty} \frac{\sin x}{x} e^{-ax}\, dx = \int_0^{\infty} \frac{\partial}{\partial a} \frac{\sin x}{x} e^{-ax}\, dx$$

$$= \int_0^{\infty} -\sin x \; e^{-ax}\, dx$$

$$= - \int_0^\infty \sin(x)\, e^{-ax}\, dx$$

let $\quad u = e^{-ax} \qquad dv = \sin(x)\, dx$

$\quad du = -a e^{-ax} dx \qquad v = -\cos(x)$

so

$$= - \left[ (-\cos(x))(e^{-ax}) - \int_0^\infty (-\cos(x))(-a e^{-ax}\, dx) \right]$$

$$= \cos(x) e^{-ax} + \int_0^\infty a \cos(x) e^{-ax}\, dx$$

let

$u = e^{-ax} \qquad dv = a \cos(x)\, dx$

$du = -a e^{-ax} dx \qquad v = a \sin(x)$

$$= \cos(x) e^{-ax} + \left[ (a \sin(x))(e^{-ax}) - \int_0^\infty (a \sin(x))(-a e^{-ax}\, dx) \right]$$

$$= \cos(x) e^{-ax} + a \sin(x) e^{-ax} + a^2 \int_0^\infty \sin(x)\, e^{-ax}\, dx$$

so

$$\frac{dI}{da} = - \int_0^\infty \sin(x)\, e^{-ax}\, dx = \cos(x) e^{-ax} + a \sin(x) e^{-ax} - a^2 \underbrace{\left( - \int_0^\infty \sin(x)\, e^{-ax}\, dx \right)}_{\frac{dI}{da}}$$

$$\frac{dI}{da} + a^2 \frac{dI}{da} = \cos(x) e^{-ax} + a \sin(x) e^{-ax}$$

$$\frac{dI}{da} \left( 1 + a^2 \right) = e^{-ax} \left( \cos(x) + a \sin(x) \right)$$

$$\frac{dI}{da} = \frac{e^{-ax}}{1 + a^2} \left( \cos(x) + a \sin(x) \right)$$

$$\frac{dI}{da} = \left[\frac{e^{-ax}}{1+a^2}\left(\cos(x) + a\sin(x)\right)\right]_0^{\infty}$$

$$= \left[\,0\,\right] - \left[\frac{1}{1+a^2}\right] = \frac{-1}{1+a^2}$$

$$\int_0^{\infty} \frac{dI}{da}\,da = \int_0^{\infty} -\frac{1}{1+a^2}\,da$$

$$I\Big]_0^{\infty} = -\tan a\Big]_0^{\infty}$$

$$I(a) = \int_0^{\infty} \frac{\sin(x)}{x} e^{-ax}\,dx$$

$$I(\infty) - I(0) = -\left[\tan a\right]_0^{\infty}$$

$$I(\infty) = 0$$

$$I(0) = \int_0^{\infty} \frac{\sin(x)}{x}\,dx$$

$$0 - \int_0^{\infty} \frac{\sin(x)}{x}\,dx = -\left(\frac{\pi}{2} - 0\right)$$

$$\int_0^{\infty} \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$$

$$\boxed{\int_{-\infty}^{+\infty} \frac{\sin(x)}{x}\,dx = \pi}$$